# Soft Effective Theory and Factorization in QED

The factorization of high- and low-energy physics we encountered in $\sigma(e^+e^- \to X)$ was especially simple since the low-energy hadronic matrix elements were local operators and the leading operator was the trivial $\mathbb{1}$ operator.

For more complicated observables, we need more sophisticated tools to analyze the low-energy contributions and these will involve nontrivial operators. The low energy parts arise when particles become soft or collinear. Indeed, we encountered singularities from such regions in the R-ratio NLO computation. To analyze this physics, one can use Soft-Collinear Effective Theory (SCET).

We'll develop this EFT in the next three lectures. Due to the presence of two different types of low-energy regions, the construction is a bit involved and to get started we'll first analyze an process in massive QED. In this example, only the soft region plays a role.

We will analyze $\sigma(e^- e^- \to e^- e^- + X_{soft})$ and will derive a factorization theorem

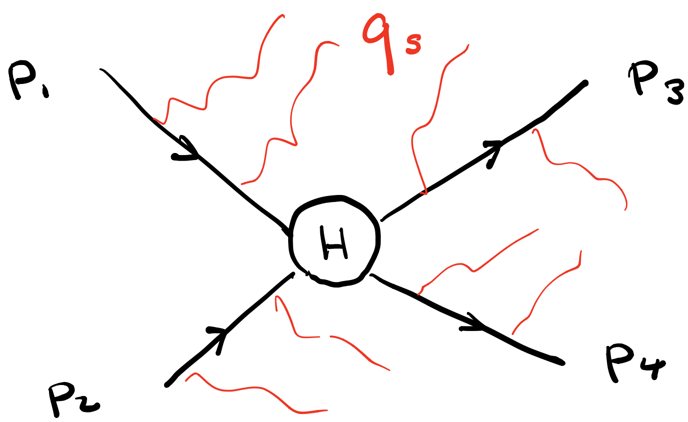
directions of charged particles

$$\sigma = \mathcal{H}[(\underbrace{m_e}_{\text{hard scales, high energy}}, \{\underline{v}\}) \, \mathcal{S}(\{\underline{v}\}, \underbrace{E_{soft}}_{\text{low energy scale}})$$

In this example, the low-energy operator will be nontrivial.

# Soft Effective Theory (see 1803.04310)

As was nicely illustrated by Holmfriður in her talk on Wednesday, one needs to include soft photons to get finite results when considering $e^- e^-$-scattering in QED




$E_s \ll m_e$

$\lambda = E_s / m_e$

How much soft radiation is included depends on the definition of the observable, but given finite detector resolution, one cannot avoid having <u>some</u> radiation.

Since the emissions have very low energy, there $e^+e^-$ will never be created. We can integrate out the $e^-$-field and use

free photons!

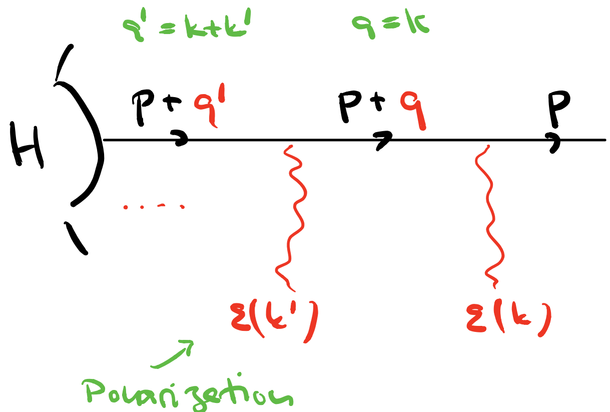
irrelevant for today, only include leading power.

$$\mathcal{L}_{\gamma}^{eff} = \mathcal{L}_{\gamma}^{(4)} + \frac{1}{m_e^2} \cancel{\mathcal{L}_{\gamma}^{(6)}} + \dots$$

$$\mathcal{L}_{\gamma}^{(4)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} .$$

This by itself is however not sufficient, since we also need to account for the $e^-$ which radiate the photons. The energy of the radiation is too small to produce $e^+e^-$ pairs, but the electrons which are present remain due to fermion number conservation. So we need a field for electrons but not positrons.

To understand what is needed, let's consider a single fermion line emitting soft photons




reference vector

Set $p^\mu = m_e v^\mu$ with $v^2 = 1$. Now expand the intermediate propagator in the soft momentum $q^\mu$:

$$\Delta(p+q) = i \frac{\not{p} + \not{q} + m_e}{(p+q)^2 - m_e^2 + i0} = i \frac{\not{p} + m_e}{2p\cdot q + i0}$$

$$= i \underbrace{\frac{\not{v}+1}{2}}_{P_v} \frac{1}{v\cdot q + i\varepsilon}$$

Note that (exercise)

$$\not{v} P_v = P_v \quad ; \quad P_v^2 = P_v \quad ; \quad P_v \not{\varepsilon} P_v = P_v \, \varepsilon\cdot v$$

Expanding the propagators and using these properties, we find that the above diagram simplifies to

$$\bar{u}(p)\, P_v \frac{i}{v\cdot q} (-ie\,\varepsilon\cdot v)\, P_v \frac{i}{v\cdot q'} (-ie\,\varepsilon'\cdot v) \cdots$$

This form of the soft amplitude is called the eikonal approximation.

Can we obtain the expanded amplitudes directly from an effective Lagrangian? (We already know the Feynman rules!) Consider

$$\mathcal{L}_{eff} = \bar{h}_v\, iv\cdot D\, h_v$$

where $h_v$ is an auxiliary fermion field which fulfils $P_v h_v = h_v$. (Can use $h_v = P_+ \psi$.)

The propagator for $h_v$ is $\dfrac{i}{v\cdot q + i\varepsilon}$ ✓

The gluon emission vertex is $-iev^\mu$ ✓

To account for the fermions along the four directions $p_i^\mu = m_e v_i^\mu$ we need four auxiliary fields

$$\rightarrow \mathcal{L}_{eff} = \sum_{i=1}^{4} \bar{h}_{v_i} i v_i \cdot D h_{v_i} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
$$+ \Delta \mathcal{L}_{int}$$

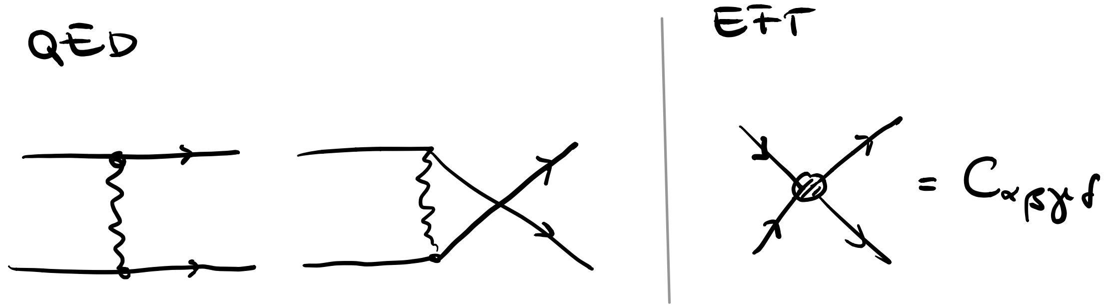
So we have split the $e^-$-field into four fields, which describe an $e^-$ along $v_i^\mu$ with momentum $m v_i^\mu + q^\mu$. The final ingredients are interactions, which take the form

$\propto \alpha/m_e^2$ Dirac index

$$\Delta \mathcal{L}_{int} = C_{\alpha\beta\gamma\delta}(v_1, v_2, v_3, v_4, m_e)\, h_{v_1}^{\alpha} h_{v_2}^{\beta} \bar{h}_{v_3}^{\gamma} h_{v_4}^{\delta}$$

at leading power in $\lambda$. (Interactions with two fields are forbidden: an $e^-$ cannot change velocity when emitting soft radiation)

To determine the Wilson coefficients we do on-shell matching and compute $e^-e^- \to e^-e^-$ w/o soft radiation




The Wilson coefficient is simply the $e^-e^-$ amplitude w/o external spinors! This works also at loop level: then both QED and the EFT have IR div's which cancel. Since the EFT diagrams vanish as $\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}$, the IR divergences in QED are in one-to-one correspondence to UV divergences of the EFT!

$\rightsquigarrow$ IR divergences can be discussed as

UV divergences in an EFT and can be

renormalized.

Now introduce the Wilson line

$$S_i(x) = \exp\left[-ie\int_{-\infty}^{0} ds\, v_i \cdot A(x+sv_i)\right]$$

which fulfils

$$v \cdot D_i\, S_i(x) = 0$$

and redefine

$$h_{v_i}(x) = S_i(x)\, h_{v_i}^{(0)}(x)$$

The fermion Lagrangian takes the form

$$\bar{h}_{v_i}\, i v_i \cdot D\, h_{v_i} = \dots = \bar{h}_{v_i}^{(0)}\, i v \cdot \partial\, h_{v_i}^{(0)} .$$

The fermion no longer interacts with the soft photons! Instead one finds Wilson lines in $\mathcal{L}_{int}$:

$$\mathcal{L}_{int} = C_{\alpha\beta\gamma\delta}\, h_{v_1}^{(0)\alpha}\, h_{v_2}^{(0)\beta}\, \bar{h}_{v_3}^{(0)\gamma}\, h_{v_4}^{(0)\delta}$$

$$\cdot\, S_{v_1} S_{v_2} S_{v_3}^{+} S_{v_4}^{+}$$

state with n soft photons

Now lets compute $\mathcal{M}(e^+e^- \to e^+e^- + X_s)$. Since there are no interactions, the amplitude factorizes

$$\mathcal{M} = \underbrace{u_{v_1}^{\alpha}\, u_{v_2}^{\beta}\, \bar{u}_{v_3}^{\gamma}\, \bar{u}_{v_4}^{\delta}\, C_{\alpha\beta\gamma\delta}}_{\mathcal{M}(e^-e^- \to e^-e^-)} \cdot$$

$$* \langle X_s | S_{v_1} S_{v_2} S_{v_3}^{+} S_{v_4}^{+} | 0 \rangle$$

Squaring the amplitude then gives a factorized cross section:

$$\boxed{\sigma = H(m_e, \{\underline{v}\}) \cdot \mathcal{S}(E_s, \{\underline{v}\})}$$

where

$$\mathcal{S} = \sum_{X_S} \left| \langle X_S | S_3^\dagger S_1 S_4^\dagger S_2 | 0 \rangle \right|^2 \Theta(E_S - E_{X_S})$$

The soft function is the low-energy matrix element, while $H = \sigma(e^- e^- \to e^- e^-)$ is the bare Wilson coefficient. We can renormalize to obtain

$$\sigma = H(m_e, \{v\}, \mu)\, \mathcal{S}(E_s, \{v\}, \mu)$$

The soft function has a very interesting property: it exponentiates

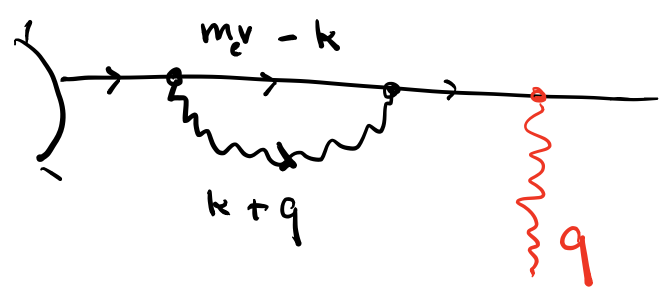
$$S(E_v, \{v\}) = \exp\left[ \frac{\alpha}{4\pi} \mathcal{S}^{(1)} \right]$$

Scale independence of $\sigma$ then implies that also the $\ln\left(\frac{\mu}{m_e}\right)$ terms in $H$ must exponentiate.

These are tied to IR divergences in on-shell amplitudes. We have thus demonstrated that the IR divergences exponentiate.

This of course assumes that our construction, which was based on expanding tree-level diagrams is also valid at the loop level. To show this, we should now discuss the "method of regions"

To discuss the expansion, let's consider the simplest example loop diagram



The associated scalar integral takes the form

$$F = \int d^d k \, \frac{1}{(k+q)^2} \frac{1}{(m_e v - k)^2 - m_e^2}$$

In the low-E theory we assume that $k_\mu \sim q_\mu \ll m_e$. Expanding the integrand yields

$$F_{low} = \int d^d k \, \frac{1}{(k+q)^2} \frac{1}{-2m_e v \cdot k} \left\{ 1 + \frac{k^2}{2m_e v \cdot k} + \dots \right\}$$

The expansion yields exactly the $i/v\cdot k$ propagators we encountered at tree level. At large $k^\mu \sim m_e$ the expansion is no longer justified and we encounter UV divergences which are stronger than in the additional integral. To correct for this consider

$$F_{high} = F - F_{low}$$

$$= \int d^d k \, \frac{1}{(k+q)^2} \left\{ \frac{1}{(m_e v - k)^2 - m_e^2} - \frac{1}{-2 m_e v\cdot k}\left[1 + \frac{k^2}{2 m v\cdot k} + \dots\right] \right\}$$

By construction, this difference in the integrand only has support for $k^\mu \gg q^\mu$ since the bracket $\{\dots\}$ vanishes for $k \to 0$. We can therefore expand the integrand around $q^\mu \to 0$. This yields

$$F_{high} = \int d^d k \, \frac{1}{k^2} \left[1 - \frac{2 q\cdot k}{k^2} + \dots\right] \{ \dots \}$$

Next we use that integrals of the form

$$\int d^d k \, (k^2)^\alpha (v\cdot k)^\beta (q\cdot k)^\gamma = 0$$

all vanish because they are scaleless. This leaves

$$F_{high} = \int d^d k \, \frac{1}{k^2} \left[1 - \frac{2 q\cdot k}{k^2} + \dots\right] \frac{1}{(m_e v - k)^2 - m_e^2} .$$

Note that this is simply the expansion of the integrand for $k^\mu \sim m_e \gg q^\mu$.

The upshot is that we recover the full integral by expanding twice. Once for

(i) $k^\mu \sim q^\mu \ll m_e$ "soft region"

$\rightsquigarrow \mathcal{F}_{\text{low}}$

(ii) $k^\mu \sim m_e \gg q^\mu$ "hard region"

$\rightsquigarrow \mathcal{F}_{\text{high}}$

The contributions (i) correspond loop integrals in the EFT (so it is OK to expand also the loop momenta!), while the contributions (ii) contribute to the matching (to get them, one can expand the full theory integrals in the small external momenta; as advertised earlier in the lecture.)

This method to obtain the expansion of an integral by expanding in different regions and integrating is very general. Sometimes, one encounters several low energy regions, e.g. "soft" + "collinear" in jet processes. One then introduces a field for each momentum region and constructs a Lagrangian which incorporates the scalings of the momenta in each case. The references given at the beginning discuss how this is done in detail.